FIN 6160
Investment Theory

Lecture 7-10

Optimal Asset Allocation

- Minimum Variance Portfolio is the portfolio with lowest possible variance. To find the optimal asset allocation for the efficient frontier with the starting point (min variance portfolio) we can set first differentiation of the variance to zero and solve for w_1.

$$\text{Variance of Portfolio, } \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}$$

$$\text{We know, } \mathrm{w}_1 + \mathrm{w}_2 = 1;$$

$$\mathrm{w}_2 = 1 - \mathrm{w}_1$$

$$\frac{d}{dw_1}[\sigma_p^2] = \frac{d}{dw_1}[w_1^2\sigma_1^2 + (1-\mathrm{w}_1)^2\sigma_2^2 + 2w_1(1-\mathrm{w}_1)\sigma_1\sigma_2\rho_{12}]$$

$$= 2w_1\sigma_1^2 + 2(1-\mathrm{w}_1)(-1)\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}[w_1\frac{d}{dw_1}(1-\mathrm{w}_1) + (1-\mathrm{w}_1)\frac{d}{dw_1}(w_1)]$$

$$= 2w_1\sigma_1^2 - 2(1-\mathrm{w}_1)\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}[w_1(-1) + (1-\mathrm{w}_1)]$$

$$= 2w_1\sigma_1^2 - 2\sigma_2^2 + 2\mathrm{w}_1\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}[1-2\mathrm{w}_1]$$

$$= 2w_1\sigma_1^2 - 2\sigma_2^2 + 2\mathrm{w}_1\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12} - 4\mathrm{w}_1\sigma_1\sigma_2\rho_{12}$$

Optimal Asset Allocation

For min variance portfolio, $\dfrac{d}{dw_1}[\sigma_p^2] = 0$

$$i.e. 2w_1\sigma_1^2 - 2\sigma_2^2 + 2w_1\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12} - 4w_1\sigma_1\sigma_2\rho_{12} = 0$$

$$or, w_1(2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_1\sigma_2\rho_{12}) = 2\sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}$$

$$or, w_1 = \frac{2\sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}{2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_1\sigma_2\rho_{12}} = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

Risk Aversion and its Implications for Portfolio Selection

- A risk-averse investor is simply one that dislikes risk (i.e. prefers less risk to more risk). Given two investment that have equal expected returns, a risk-averse investor will choose the one with less risk.
- A risk-seeking (risk-loving) investor actually prefers more risk to less and, given equal expected returns, will choose the more risky investment. A risk-neutral investor has no preference regarding risk and would be indifferent between two such investments.
- Consider this gamble: A coin will be flipped; if it comes up head, you receive $100; if it comes up tails, you receive nothing. The expected payoff is 0.5($100)+0.5($0)=$50. A risk averse investor would choose a payment of $50 (a certain outcome) over the gamble. A risk-seeking investor would prefer the gamble to a certain payment of $50. A risk-neutral investor would be indifferent between the gamble and a certain payment of $50.
- If expected returns are identical, a risk-averse investor will always choose the investment with the least risk. However, an investor may select a very risky portfolio despite being risk-averse; a risk-averse investor will hold very risky assets if he feels that the extra return he expects to earn is adequate compensation for the additional risk.

Risk Aversion and its Implications for Portfolio Selection

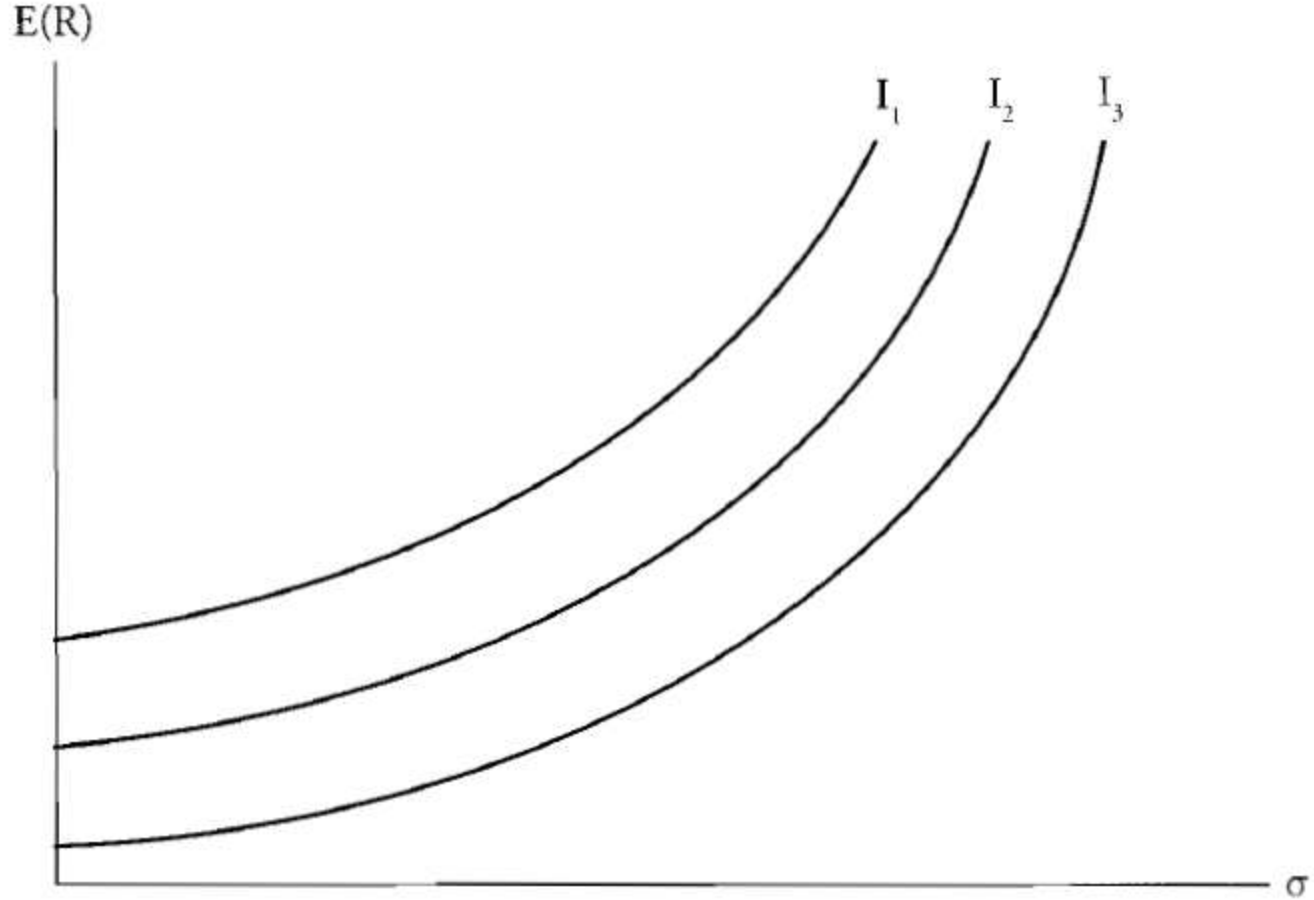
- An investor's utility function represents the investor's preferences in terms of risk and return (i.e. his degree of risk aversion). Investors maximize their utility function for various investments.

$$U = E(R) - a\sigma_R^2$$

- The parameter "a" above measures the degree of risk aversion. The larger "a", the more risk averse the investor is.
- An indifference curve plots combinations of risk and expected return among which an investor is indifferent. In constructing indifference curves for portfolios based on only their expected return and standard deviation of returns, we are assuming that these are the only portfolio characteristics that investors care about.
- The investor's expected utility is the same for all points along a single indifference curve. Indifference curves slope upward for risk-averse investors because they will only take on more risk if they are compensated with greater expected returns. An investor who is relatively more risk-averse requires a relatively greater increase in expected return to compensate for a given increase in risk. In other words, a more risk averse investor will have steeper indifference curves.

Utility Function to Measure the degree of Risk

Figure 4: Risk-Averse Investor's Indifference Curves



Diversification with Risk-less Assets

- We have learned to construct portfolios with risky assets.
- An investor can alternatively combine a risky investment with an investment in a riskless or risk-free asset security such as Treasury bills.

	Merville Company	Risk-free Asset
Expected Return	14%	10%
Standard Dev.	0.2	0

- Suppose an investor chooses to invest \$350 on Merville and \$650 on risk-free asset.

$$\text{Expected Return, } \mathrm{E}(\mathrm{r}_p) = w_1\mathrm{E}(\mathrm{r}_1) + w_2\mathrm{E}(\mathrm{r}_2) = 0.35x0.14 + 0.65x0.1 = 0.114 = 11.4\%$$

- Risk-free asset has no variance, thus both covariance between the assets and variance of risk free asset is zero.

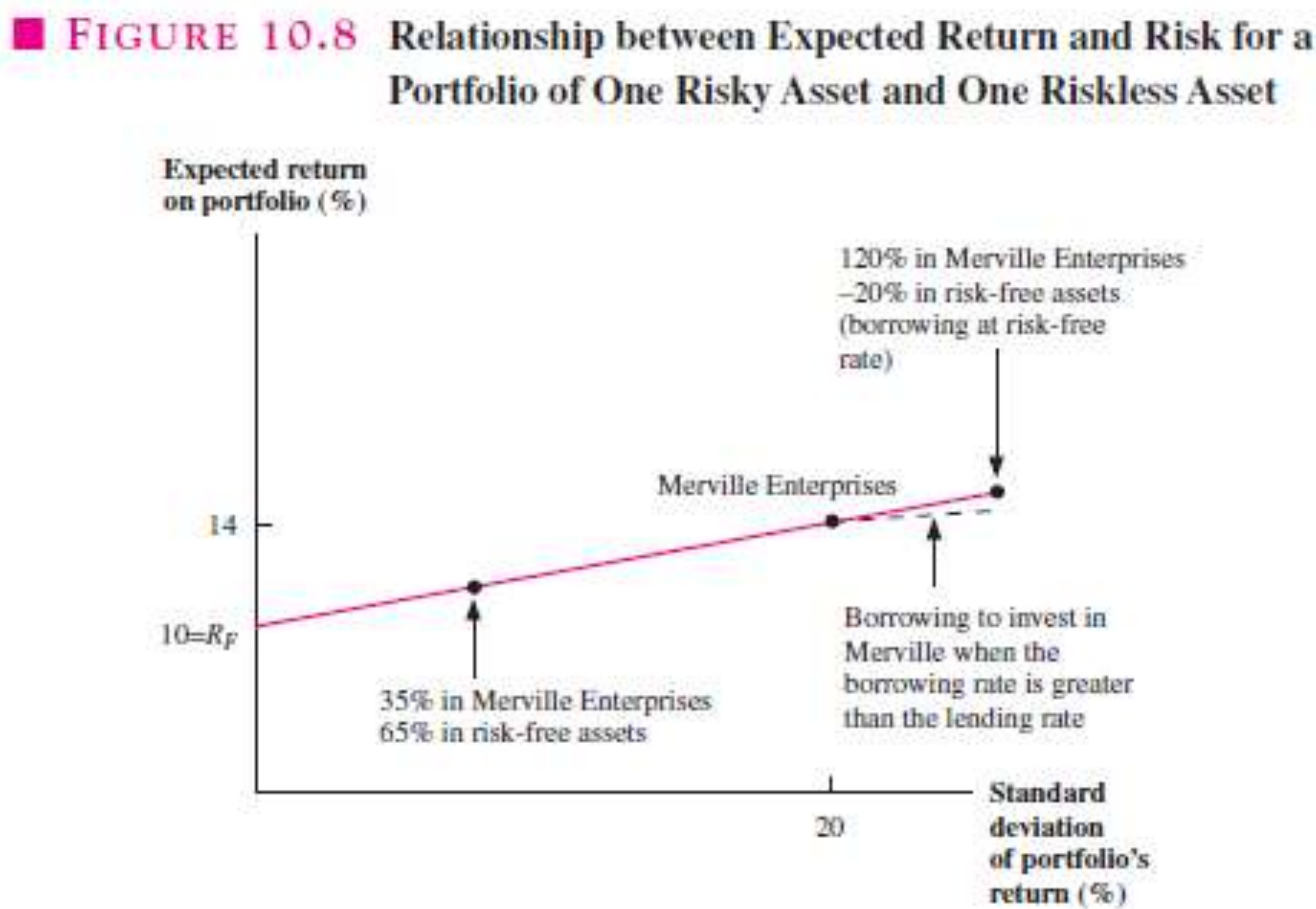
$$\text{Variance of Portfolio, } \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

$$\text{Here, } \sigma_2^2 = \sigma_{12} = 0$$

$$\text{i.e. } \sigma_p^2 = w_1^2\sigma_1^2$$

$$or, \sigma_p = w_1\sigma_1 = 0.35x0.2 = 0.07$$

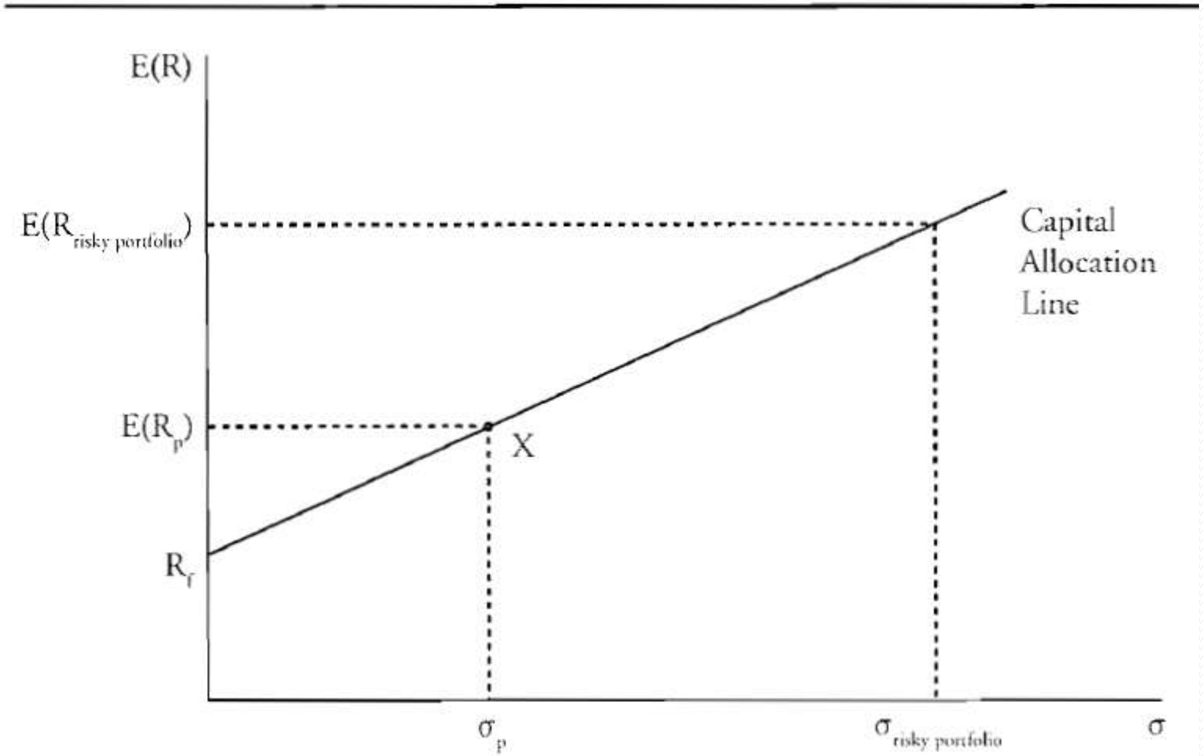
Diversification with Risk-less Assets



Diversification with Risk-less Assets

- Combining a risky portfolio with a risk-free asset is the process that supports the two fund separation theorem, which states that all investors' optimum portfolios will be made up of some combination of an optimal portfolio of risky assets and the risk free asset. The line representing these possible combinations of risk-free assets and the optimal risky asset portfolio is referred to as the capital allocation line (CAL).

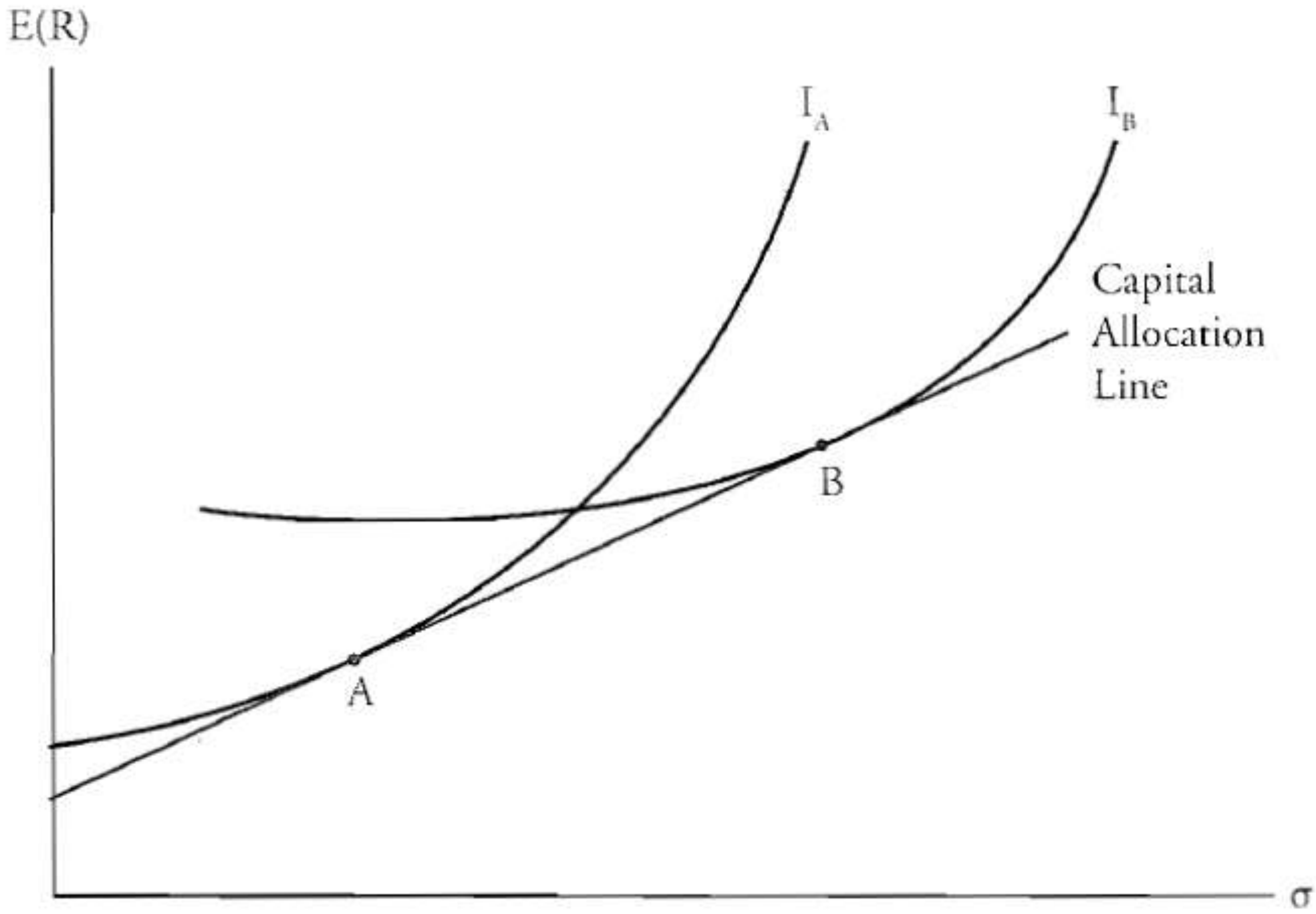
Figure 5: Capital Allocation Line and Risky Asset Weights



▪Investors who are less risk-averse will select portfolios that are more risky. We know that the less an investor's risk aversion, the flatter his indifference curves. As in the figure, the flatter indifference curve for investor B (I_B) results in an optimal (tangency) portfolio that lies to the right of the one that results from a steeper indifference curve, such as that for investor A (I_A). An investor who is less risk averse should optimally choose a portfolio with more invested in the risky asset portfolio and less invested in the risk-free asset.

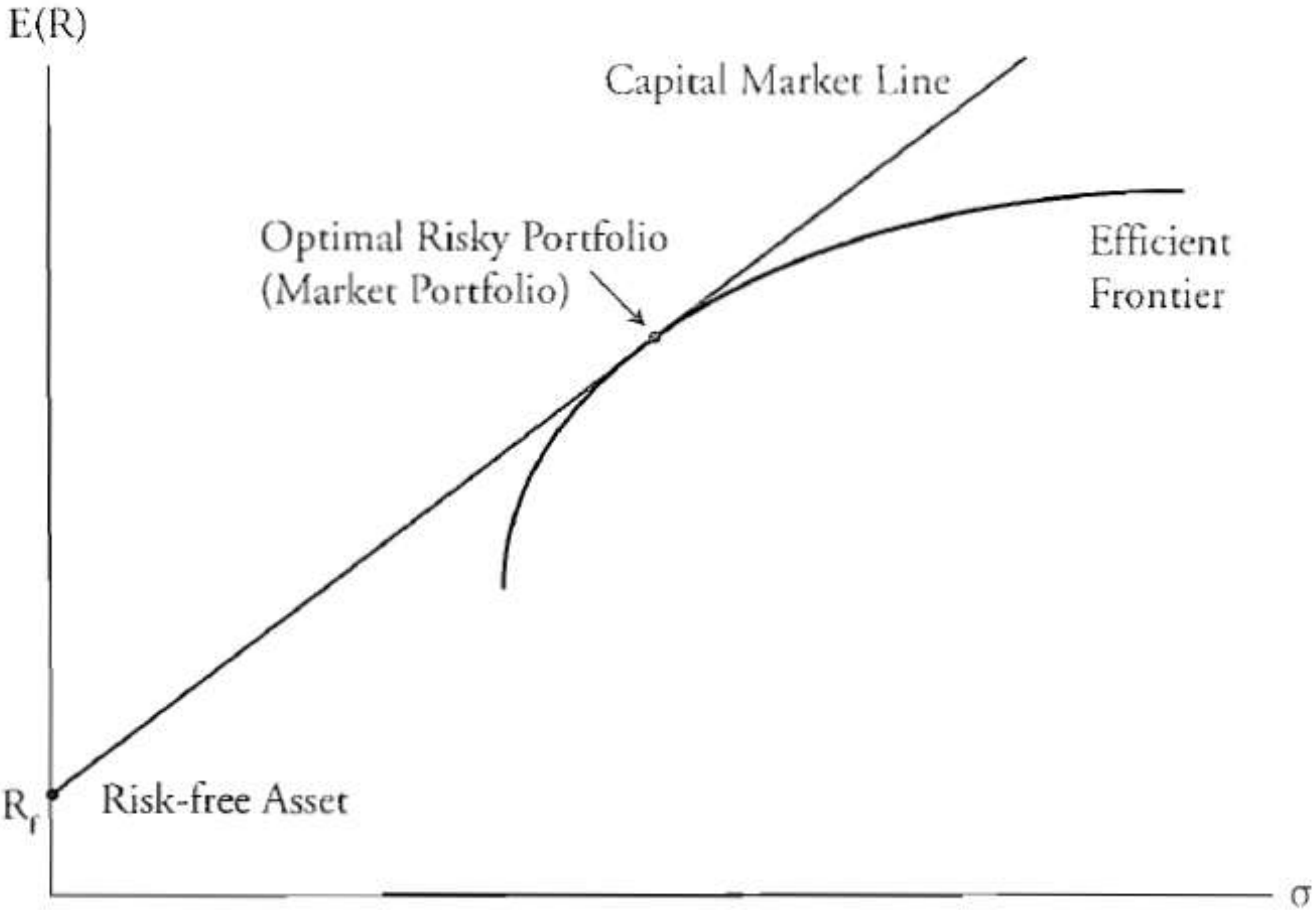
Diversification with Risk-less Assets

Figure 7: Portfolio Choices Based on Investor's Indifference Curves

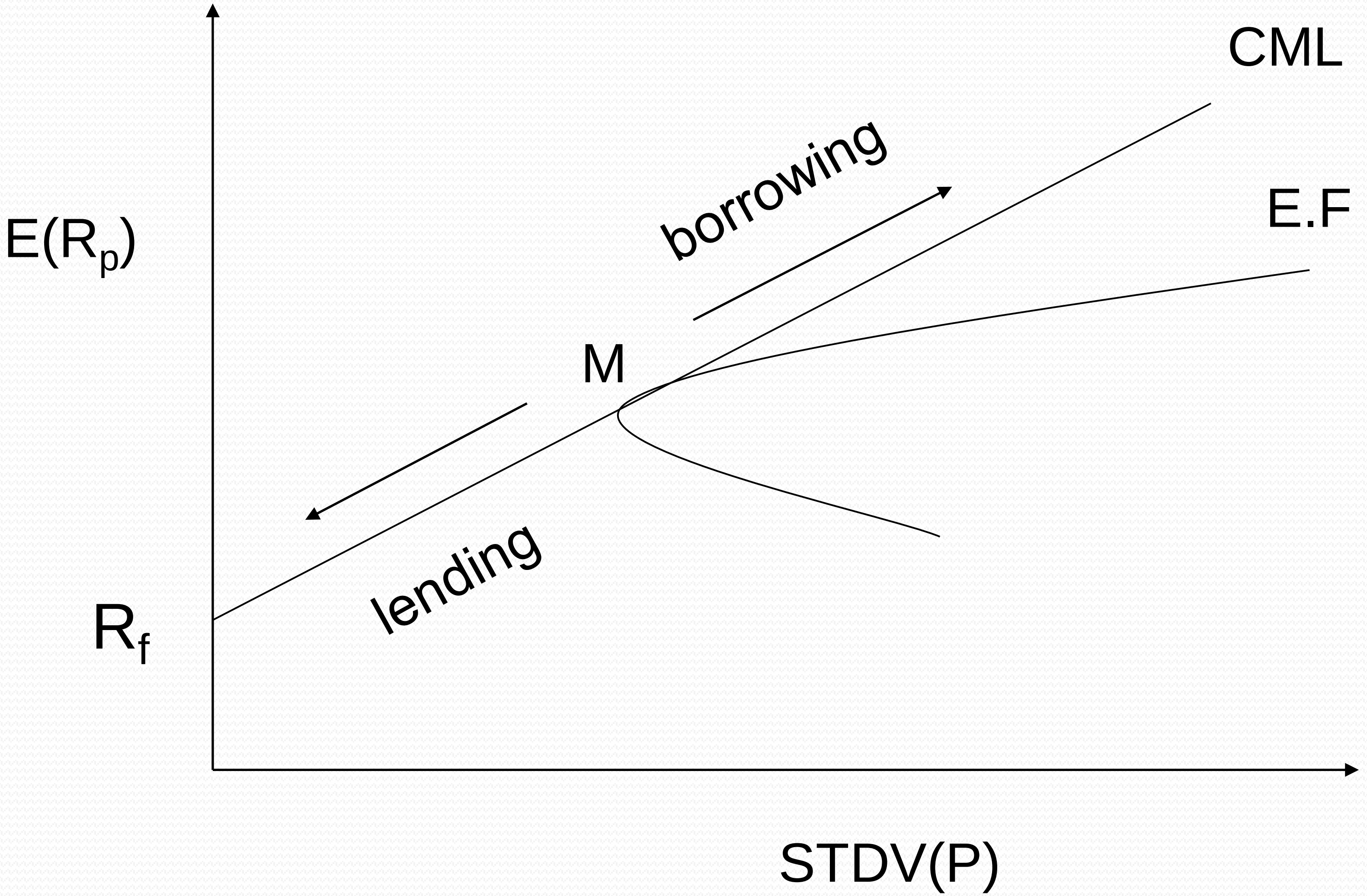


- A simplifying assumption underlying modern portfolio theory and the capital asset pricing model is that investors have homogeneous expectations (i.e. they all have the same estimates of risk, return, and correlations with other risky assets for all risky assets). Under this assumption, all investors face the same efficient frontier of risky portfolios and will all have the same optimal risky portfolio and the optimal CAL (capital market line-CML) for any investor is the one that is just tangent to the efficient frontier.
- The tangent point of the CML and efficient frontier is market portfolio that gives maximum return for a given risk. CML states that an optimal investment strategy is to split capital between risk-free asset and the market portfolio. Market portfolio is a market value weighted portfolio of all existing securities.
- In the mean-variance world, a risk averse investor has to decide what portion of his wealth he should invest in an optimally formed (efficient) portfolio of risky assets (e.g. mutual fund) and what portion in the risk free asset.

Figure 3: Determining the Optimal Risky Portfolio and Optimal CAL Assuming Homogeneous Expectations



Efficient Frontier vs. CML



▪Along CML, expected portfolio return is a linear function of portfolio risk. The equation of this line is as follows:

$$\text{Expected Return, } \mathrm{E}(\mathrm{r}_p) = R_f + [\frac{\mathrm{E}(\mathrm{R}_m)\text{-}R_f}{\sigma_m}]\sigma_p$$

▪The difference between the expected return on the market and the risk free rate is termed the market risk premium. We can rewrite the CML equation as:

$$\text{Expected Return, } \mathrm{E}(\mathrm{r}_p) = R_f + [\mathrm{E}(\mathrm{R}_m)\text{-}R_f]\frac{\sigma_p}{\sigma_m}$$

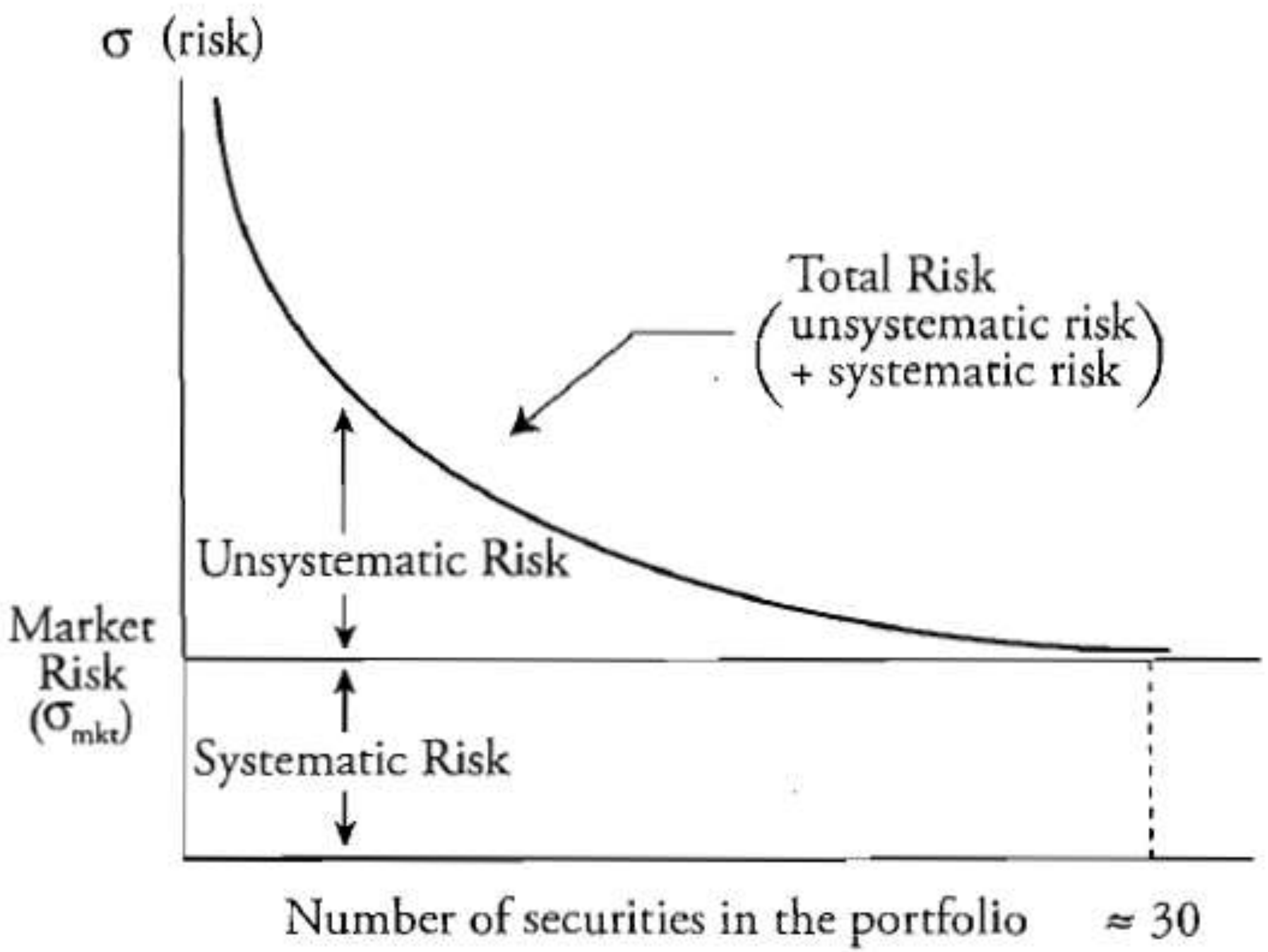
▪An investor can expect to get one unit of market risk premium above the risk free rate for every unit of market risk that the investor is willing to accept.

Systematic and Non-systematic Risk

- When an investor diversifies across assets that are not perfectly correlated, the portfolio's risk is less than the weighted average of the risks of the individual securities in the portfolio. The risk that is eliminated by diversification is called unsystematic risk (also called unique, diversifiable, or firm specific risk). Since the market portfolio contains all the risky assets, it must be well a diversified portfolio. The risk that remains and can not be diversified away, is called the systematic risk (also called non-diversifiable or market risk).
- The concept of systematic risk applies to individual securities as well as to portfolios. Some securities' returns are highly correlated with overall market returns. Examples of firms that are highly correlated with market returns are luxury goods manufactures such as Ferrari Automobiles and Harley Davidson Motorcycles. These firms have high systematic risk (i.e they are very responsive to market).
- Other firms, such as utility companies, respond very little to changes in the systematic risk factors. These firms have very little systematic risk. Hence total risk (measured by standard deviation) can be broken down into its component parts: unsystematic risk and systematic risk.

Systematic and Non-systematic Risk

Figure 4: Risk vs. Number of Portfolio Assets



Multi-factor model vs Single Factor or Single Index Model

▪Multi-factor models to predict expected returns commonly use macro economic factors such as GDP growth, inflation, consumer confidence, along with fundamental factors such as earnings, earnings growth, firm size and research expenditures. The general form of a multi factor model with k factors is as follows:

$$\text{Expected Return, } \mathrm{E}(\mathrm{r}_i) = R_f + \beta_1 \mathrm{E}(\text{factor}_1) + \beta_2 \mathrm{E}(\text{factor}_2) + \ldots\ldots + \beta_k \mathrm{E}(\text{factor}_k)$$

▪Single index model in contrast is a single factor or market model. The only factor is the expected excess return on the market portfolio (market index). The form of the single index model is as follows:

$$\text{Expected Return, } \mathrm{E}(\mathrm{r}_i) = R_f + \beta_i [\mathrm{E}(\mathrm{r}_m) - R_f]$$

$$or, \mathrm{E}(\mathrm{r}_i) - R_f = \beta_i [\mathrm{E}(\mathrm{r}_m) - R_f]$$

▪In this case, the beta for asset i is a measure of how sensitive the excess return on asset i is to the excess return on the overall market portfolio.

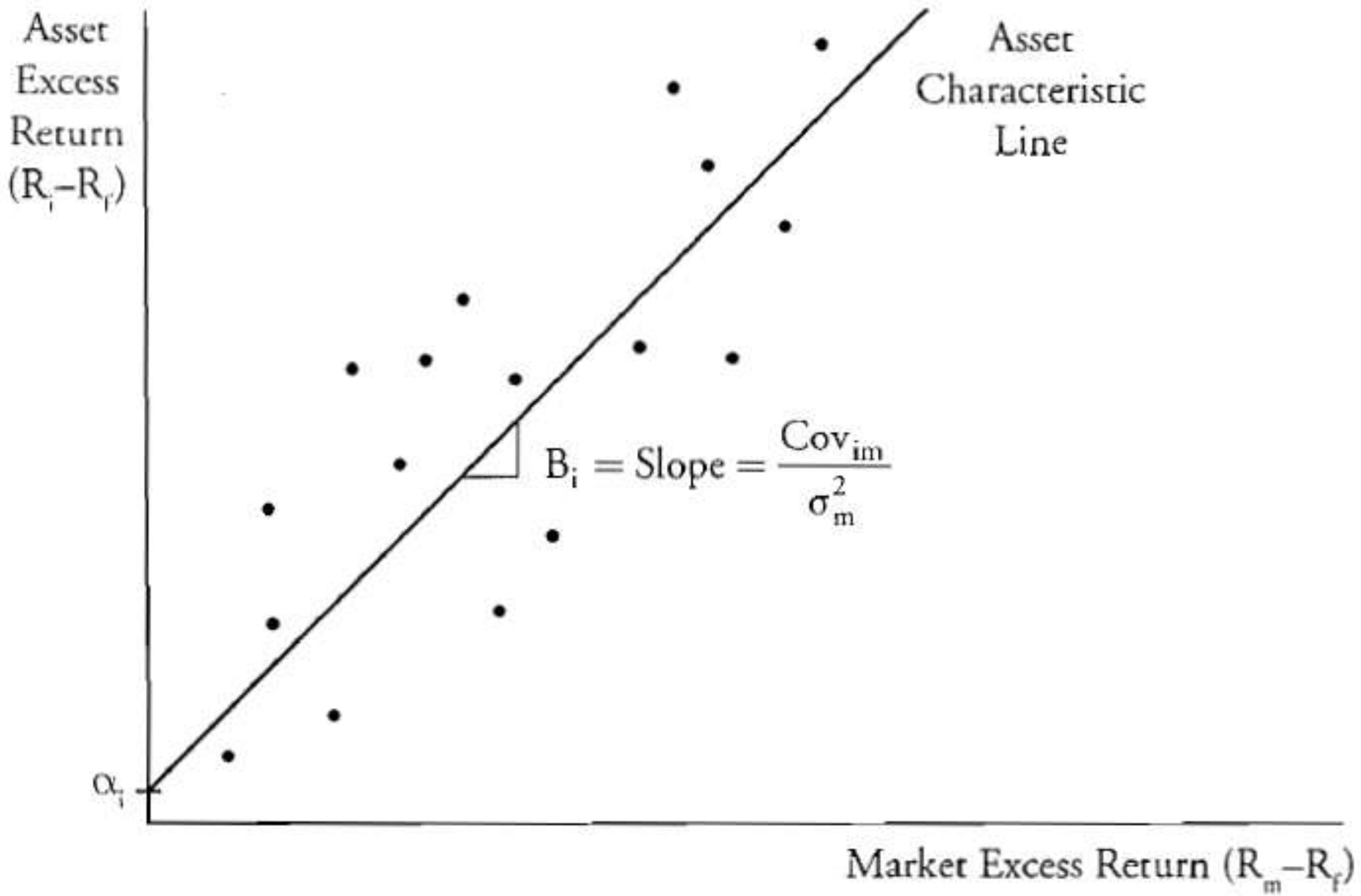
Calculate and Interpret Beta

- The sensitivity of an asset's return to the return on the market index in the context of the market model is referred to as its beta. Beta measures the responsiveness of a security to movements in the market portfolio.
- The contribution of a security to the variance of a diversified portfolio is best measured by beta. Therefore, beta is the proper measure of the risk of an individual security for a diversified investor.
- Beta measures the systematic risk of a security. Thus, diversified investors pay attention to the systematic risk of each security. However, they ignore the unsystematic risk of individual securities, since unsystematic risks are diversified away in a large portfolio

$$\beta_i = \frac{Cov(i,m)}{\sigma_m^2} = \frac{\rho(i,m)\sigma_i\sigma_m}{\sigma_m^2} = \rho(i,m)\frac{\sigma_i}{\sigma_m}$$

- One useful property is that the average beta across all securities, when weighted by the proportion of each security's market value to that of the market portfolio, is 1. That is, the beta of the market portfolio is 1. For aggressive securities beta>1; for defensive securities beta<1 and for neutral securities beta=1

Figure 5: Regression of Asset Excess Returns against Market Asset Returns



Relationship between Risk and Expected Return

▪Capital Asset Pricing Model (CAPM) implies that the expected return on a security is linearly related to its beta. Because the average return on the market has been higher than the average risk free rate over long periods of time, market premium is presumably positive. Thus the formula implies that the expected return on a security is positively related to its beta.

Capital Asset Pricing Model (CAPM):

$$E(R_i) = R_f + \beta_i \times [E(R_m) - R_f]$$

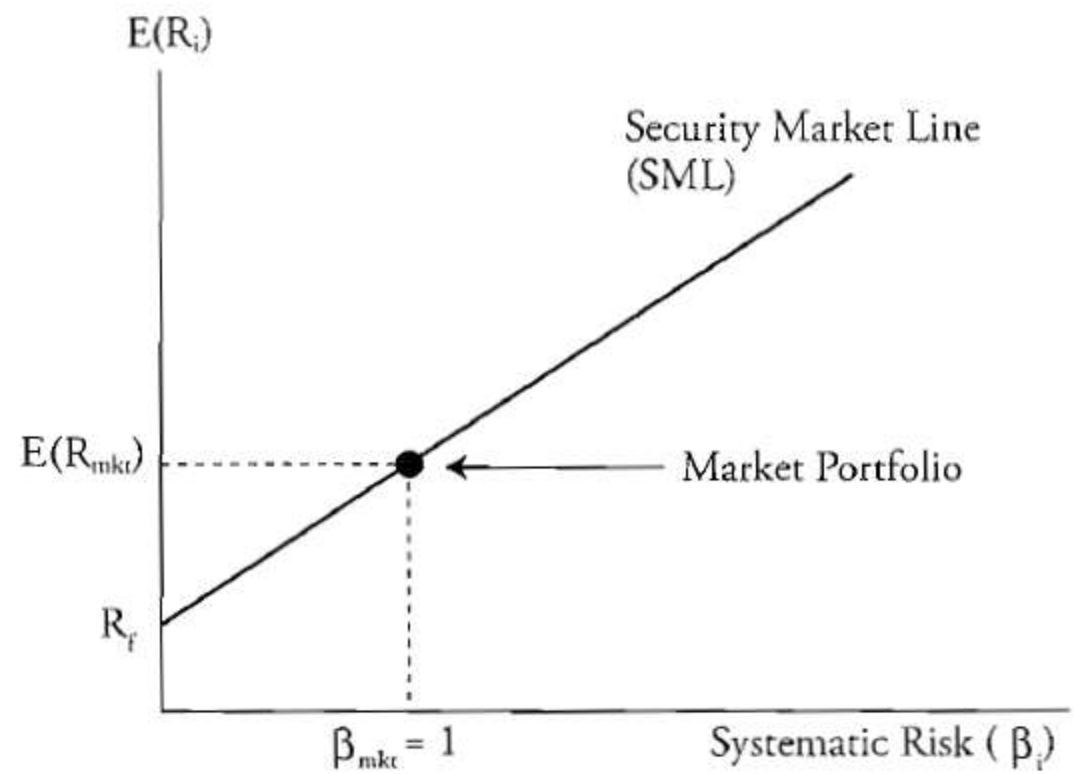
Expected Return on a Security	=	Risk-free rate	+	Beta of the security	x	Market Risk Premium

▪If β=0, $E(R_i) = R_f$ that is, the expected return on the security is equal to the risk free rate. Because a security with zero beta has no relevant risk, its expected return should equal the risk-free rate.

▪If β=1, $E(R_i) = E(R_m)$ that is, the expected return on the security is equal to the expected return on the market. This makes sense because the beta of the market portfolio is also 1.

Relationship between Risk and Expected Return

Figure 7: The Capital Asset Pricing Model



- Security Market Line (SML) is the graphical depiction of the capital asset pricing model (CAPM)

- The CML uses total risk σ on the X-axis. Hence only efficient portfolios will plot on the CML. On the other hand, the SML uses beta (systematic risk) on the X-axis. So in a CAPM world, all properly priced securities and portfolios of securities will plot on the SML.

CAPM

- **Example**: The expected return on the market is 15%, the risk free rate is 8%, and the beta for the stock A is 1.2. Compute the rate of return that would be expected (required) on this stock.
- Answer: $E(R_A) = 0.08 + 1.2(0.15 - 0.08) = 0.164 = 16.4\%;$ Here, $\beta_A > 1;\ E(R_A) > E(R_M)$
- **Example**: The expected return on the market is 15%, the risk free rate is 8%, and the beta for the stock B is 0.8. Compute the rate of return that would be expected (required) on this stock.
- Answer: $E(R_B) = 0.08 + 0.8(0.15 - 0.08) = 0.136 = 13.6\%;$ Here, $\beta_B < 1;\ E(R_B) < E(R_M)$
- **Example**: Acme, Inc., has a capital structure that is 40% debt and 60% equity. The expected return on the market is 12%, and the risk free rate is 4%. What discount rate should an analyst use to calculate the NPV of a project with an equity beta of 0.9 if the firm's after tax cost of debt is 5%
- Answer: The required return on equity for this project is, 0.04+0.9(0.12-0.04)=0.112=11.2%
- The appropriate discount rate is a weighted average of the costs of debt and equity for this project, 0.40x0.05+0.6x0.112=0.0872=8.72%

Forecast Data

Stock	*Price Today*	*E(Price) in 1 Year*	*E(Dividend) in 1 Year*	*Beta*
A	$25	$27	$1.00	1.0
B	40	45	2.00	0.8
C	15	17	0.50	1.2

- Suppose risk free rate and market return are 7% and 15% respectively. Compute the expected and required return on each stock, determine whether each stock is undervalued, overvalued or properly valued an outline an appropriate trading strategy.

Stock	*Forecast Return*	*Required Return*
A	($27 – $25 + $1) / $25 = 12.0%	0.07 + (1.0)(0.15 – 0.07) = 15.0%
B	($45 – $40 + $2) / $40 = 17.5%	0.07 + (0.8)(0.15 – 0.07) = 13.4%
C	($17 – $15 + $0.5) / $15 = 16.6%	0.07 + (1.2)(0.15 – 0.07) = 16.6%

Stock A is *overvalued.* It is expected to earn 12%, but based on its systematic risk it should earn 15%. It plots *below* the SML.
Stock B is *undervalued.* It is expected to earn 17.5%, but based on its systematic risk it should earn 13.4%. It plots *above* the SML.
Stock C is *properly valued.* It is expected to earn 16.6%, and based on its systematic risk it should earn 16.6%. It plots *on* the SML.

The appropriate trading strategy is:

- Short sell Stock A.
- Buy Stock B.
- Buy, sell, or ignore Stock C.

Assumptions of CAPM

- Investors make their investment decisions according to mean-variance rule
- Investors can borrow or lend at the risk-free rate.
- No transactions cost for diversification
- Investors are price takers and have homogeneous expectations or beliefs regarding future expected returns, variances and covariance.
- All assets are marketable and perfectly divisible.
- There are no market imperfections such as taxes, regulations, or restrictions on short selling.

Portfolio Performance Evaluation

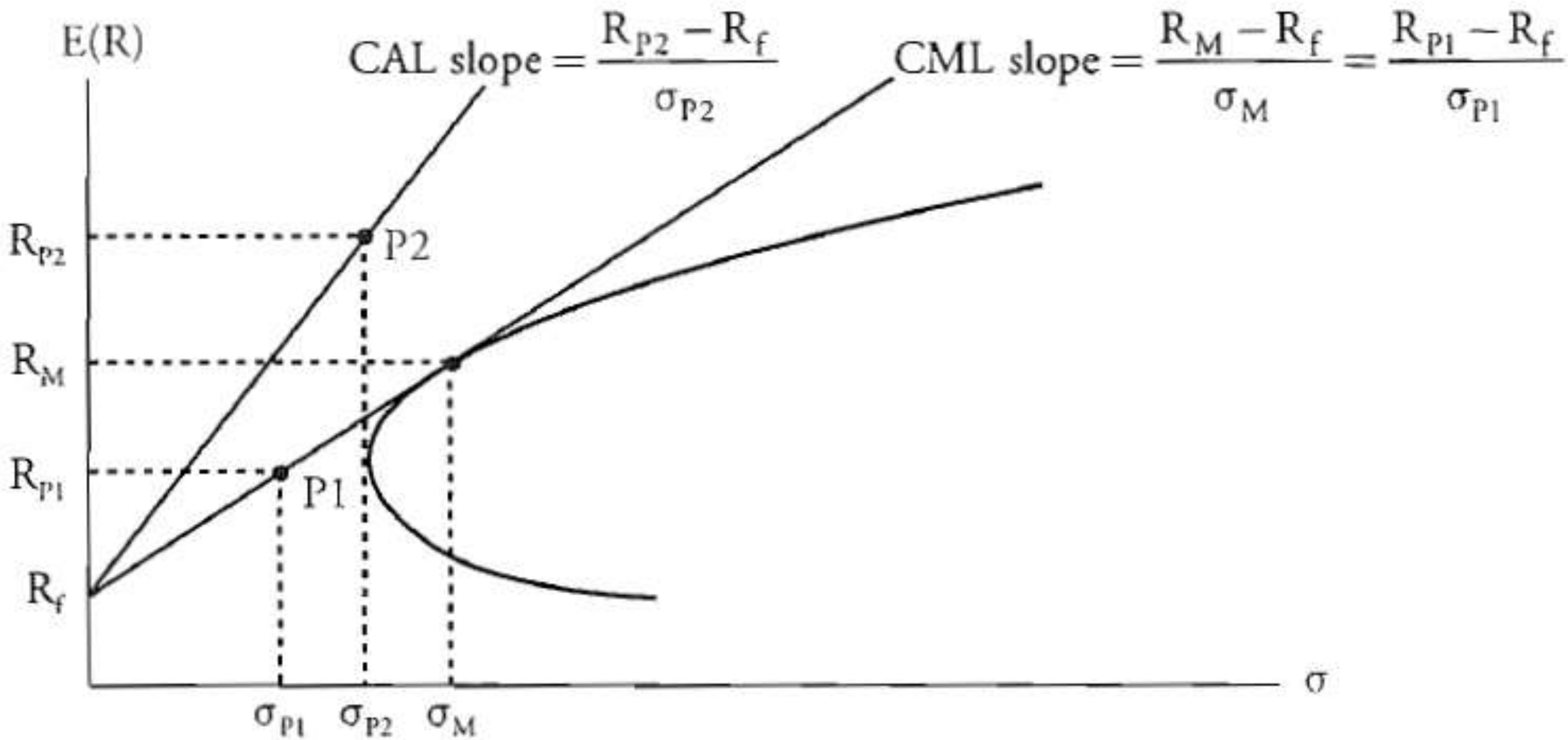
- When we evaluate the performance of a portfolio with risk that differs from that of a benchmark, we need to adjust the portfolio returns for the risk of the portfolio. There are several measures of risk-adjusted returns that are used to evaluate relative portfolio performance.

- **Sharpe Ratio**: The Sharpe Ratio (SR) of a portfolio is the ratio of the risk-premium to the portfolio`s standard deviation, and higher Sharpe ratios indicate better risk-adjusted portfolio performance. It tells us the extra return we would get from a portfolio to extra unit of STDV. Sharpe ratio uses total risk, rather than systematic risk, it accounts for any unsystematic that the portfolio manager has taken. The value of the Sharpe ratio is only useful for comparison with the Sharpe ratio of another portfolio. Sharpe ratio is a slope measure and are same for all portfolios along the CML

$$\text{Sharpe Ratio} = \frac{E(R_p) - r_f}{\sigma_p}$$

Portfolio Performance Evaluation

Figure 9: Sharpe Ratios as Slopes



Example 1-Sharpe ratio

- **Stock A Stock B Risk Free rate**
- **Average ret:**
- **7% 15% 2%**

  StDev ret.
- **8% 1.96%**
- **Covariance ret.**
- **0.0011**
- **Percentage of stock A =30% stock B = 70%**
- **Exp. Port. Ret. = 12.60%**
- **Port. staDev = 10.32%**
- **Excess ret. = 10.60% ---→ Sharpe ratio = 1.0271**

Example 2-Sharpe ratio

- Stock A Stock B Risk Free rate
- Average ret.
- 7% 15% 2%

  StDev ret.
- 8%% 1.96%
- Covariance ret.
- 0.0011
- Percentage of stock A =40% stock B = 60%
- Exp. Port. Ret. = 11.80%
- Port. staDev = 9.28%
- Excess ret. = 9.80% ---→ Sharpe ratio = 1.0557
- Portfolio 2 is the best combination of risky assets available

Portfolio Performance Evaluation

▪**M-squared**: The M-squared measure produces the same portfolio rankings as the Sharpe ratio but is stated in percentage terms. It is calculated as:

$$M - squared = (R_p - R_f)\frac{\sigma_M}{\sigma_p} - (R_M - R_f)$$

▪**Treynor Measure and Jensen's Alpha**: Treynor Measure and Jensen's Alpha is based on systematic risk (beta) rather than total risk.

$$\text{Treynor Measure} = \frac{(R_p - R_f)}{\beta_p}$$

$$\text{Jensen's Alpha,}\alpha_p = (R_p - R_f) - \beta_p(R_M - R_f)$$

▪Under the assumptions of the CAPM we expect Jensen's Alpha α=0. We get an estimate of this alpha from standard regression analysis but we need to take into account the statistical significance of the alpha. A positive and statistically significant Jensen's alpha implies outperformance. Jensen's alpha is a measure of managerial ability because it shows whether the manager had added any value over and above the return implied by the beta risk he had undertaken.

Passive vs Active Portfolio Management

- Investors who believe market prices are informationally efficient often follow a passive investment strategy (i.e. invest in an index of risky assets that serves as a proxy for the market portfolio and allocate a portion of their investable assets to a risk-free asset, such as short term govt securities)

- In practice, many investors and portfolio managers believe their estimates of security values are correct and market prices are incorrect. Such investors will not use the weights of the market portfolio but will invest more than the market weights in securities that they believe are undervalued and less than the market weights in securities which they believe are overvalued. This is referred to as active portfolio management to differentiate it from a passive investment strategy that utilizes a market index for the optimal risky asset portfolio.